

# Several applications of Cartwright-Field's inequality

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**Abstract.** In this paper we present several applications of Cartwright-Field's inequality. Among these we found Young's inequality, Bernoulli's inequality, the inequality between the weighted power means, Hölder's inequality and Cauchy's inequality. We give also two applications related to arithmetic functions and to operator inequalities.

**Keywords :** Cartwright-Field's inequality, Young's inequality, Bernoulli's inequality, Hölder's inequality, arithmetic function, operator inequality

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## 1 INTRODUCTION

An important result related to the improvement of the inequality between arithmetic and geometric means (AM-GM) was obtained by D. I. Cartwright and M. J. Field in [2], which is given in the following way: if  $0 < m = \min\{x_1, \dots, x_n\}$  and  $M = \max\{x_1, \dots, x_n\}$ , then

$$\begin{aligned} \frac{1}{2M} \sum_{i=1}^n \alpha_i \left( x_i - \sum_{k=1}^n \alpha_k x_k \right)^2 &\leq \sum_{i=1}^n \alpha_i x_i - \prod_{i=1}^n x_i^{\alpha_i} \leq \\ &\leq \frac{1}{2m} \sum_{i=1}^n \alpha_i \left( x_i - \sum_{k=1}^n \alpha_k x_k \right)^2, \end{aligned} \quad (1.1)$$

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where  $\alpha_i > 0$  for all  $i = 1 \dots n$  and  $\alpha_1 + \dots + \alpha_n = 1$ . For  $n = 2$ , this inequality may be written as follows:

$$\frac{\lambda(1-\lambda)}{2M}(a-b)^2 \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq \frac{\lambda(1-\lambda)}{2m}(a-b)^2, \quad (1.2)$$

where  $a, b > 0$ ,  $m = \min\{a, b\}$ ,  $M = \max\{a, b\}$  and  $\lambda \in [0, 1]$ . Since  $\frac{\lambda(1-\lambda)}{2M}(a-b)^2 \geq 0$ , we deduce Young's inequality (see [6, 9])

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b \quad (1.3)$$

Therefore, inequality (1.2) is an improvement of Young's inequality and at the same time gives a reverse inequality for the inequality of Young.

In [4], we presented two inequalities which give two different reverse inequalities for the Young's inequality, namely:

$$0 \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq a^\lambda b^{1-\lambda} \exp \left\{ \frac{\lambda(1-\lambda)(a-b)^2}{m^2} \right\} - a^\lambda b^{1-\lambda} \quad (1.4)$$

and

$$0 \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq \lambda(1-\lambda) \left\{ \log \left( \frac{a}{b} \right) \right\}^2 M, \quad (1.5)$$

where  $a, b > 0$ ,  $m \equiv \min\{a, b\}$  and  $M = \max\{a, b\}$  and  $\lambda \in [0, 1]$ .

**Remark 1.1.** *The first inequality of (1.2) clearly gives an improvement of the first inequality in (1.4) and (1.5). For  $0 < a, b < 1$ , we find the right hand side of the second inequality of (1.2) gives tighter upper bound than that of (1.5), from the inequality  $\frac{x-y}{\log x - \log y} < \frac{x+y}{2}$ , for  $x, y > 0$ . For  $a, b > 1$ , we find the right hand side of the second inequality of (1.5) gives tighter upper bound than that of (1.2), from the inequality  $\sqrt{xy} < \frac{x-y}{\log x - \log y}$ , for  $x, y > 0$ . In addition, we find the right hand side of the second inequality of (1.2) gives tighter upper bound than that of (1.4) for  $a, b > 0$ , from  $e^x > 1 + x$ .*

Remark 1.1 supports the importance to study the inequality (1.2) for several applications which will be given in the following sections.

## 2 MAIN APPLICATIONS

**Lemma 2.1.** *For  $x > -1$  and  $\lambda \in [0, 1]$  there is the following inequality*

$$\frac{\lambda(1-\lambda)}{2M}x^2 \leq \lambda x + 1 - (x+1)^\lambda \leq \frac{\lambda(1-\lambda)}{2m}x^2, \quad (2.1)$$

where  $m = \min\{x+1, 1\}$  and  $M = \max\{x+1, 1\}$ .

*Proof.* By replacing  $\frac{a}{b}$  of  $t$  in inequality (1.2) we obtain the inequality

$$\frac{\lambda(1-\lambda)}{2M}(t-1)^2 \leq \lambda t + 1 - \lambda - t^\lambda \leq \frac{\lambda(1-\lambda)}{2m}(t-1)^2, \quad (2.2)$$

for all  $t > 0$  and  $\lambda \in [0, 1]$ , where  $m = \min\{t, 1\}$  and  $M = \max\{t, 1\}$ . Substituting  $t = x + 1$  in inequality (2.2), we find the inequality desired.  $\square$

**Remark 2.1.** *Inequality (2.1) refines the inequality of Bernoulli, namely, for  $x > -1$  and  $\lambda \in [0, 1]$ , we have*

$$\lambda x + 1 \geq (x + 1)^\lambda \quad (2.3)$$

because  $\frac{\lambda(1-\lambda)}{2M}x^2 \geq 0$  in inequality (2.1).

Next we will establish a refinement of the inequality between the weighted power means, based on inequality (2.1).

**Theorem 2.2.** *If  $a_i > 0$ ,  $p_i > 0$ ,  $i = 1 \dots n$ ,  $0 < r \leq s$ ,  $M_r(a, p) = \left( \frac{\sum_{i=1}^n p_i a_i^r}{\sum_{i=1}^n p_i} \right)^{1/r}$  and  $M_s(a, p) = \left( \frac{\sum_{i=1}^n p_i a_i^s}{\sum_{i=1}^n p_i} \right)^{1/s}$ , then there is the inequality*

$$\frac{A}{M} \leq [M_s(a, p)]^r - [M_r(a, p)]^r \leq \frac{A}{m}, \quad (2.4)$$

where

$$A = \frac{r(s-r)}{2s^2} [M_s(a, p)]^r \cdot \frac{\sum_{i=1}^n p_i \left( \frac{a_i^s}{[M_s(a, p)]^s} - 1 \right)^2}{\sum_{i=1}^n p_i},$$

$$m = \min_{i=1, n} \left\{ \frac{a_i^s}{[M_s(a, p)]^s}, 1 \right\} \text{ and } M = \max_{i=1, n} \left\{ \frac{a_i^s}{[M_s(a, p)]^s}, 1 \right\}.$$

*Proof.* If  $r = s$ , then we have the equality in relation (2.4). Let  $r < s$ . In inequality (2.2) we

consider  $t = \frac{a_i^s}{[M_s(a, p)]^s}$  and  $\lambda = \frac{r}{s} < 1$ , thus, we deduce the inequality

$$\begin{aligned} \frac{r(s-r)}{2s^2M} \left( \frac{a_i^s}{[M_s(a, p)]^s} - 1 \right)^2 &\leq \frac{r}{s} \frac{a_i^s}{[M_s(a, p)]^s} + 1 - \frac{r}{s} - \frac{a_i^r}{[M_s(a, p)]^r} \\ &\leq \frac{r(s-r)}{2s^2m} \left( \frac{a_i^s}{[M_s(a, p)]^s} - 1 \right)^2. \end{aligned} \quad (2.5)$$

Multiplying by  $p_i$  in inequality (2.5) and taking the sum for  $i = 1 \dots n$ , we obtain the following inequality

$$\begin{aligned} \frac{r(s-r)}{2s^2M} \frac{\sum_{i=1}^n p_i \left( \frac{a_i^s}{[M_s(a, p)]^s} - 1 \right)^2}{\sum_{i=1}^n p_i} &\leq 1 - \left[ \frac{M_r(a, p)}{M_s(a, p)} \right]^r \\ &\leq \frac{r(s-r)}{2s^2m} \frac{\sum_{i=1}^n p_i \left( \frac{a_i^s}{[M_s(a, p)]^s} - 1 \right)^2}{\sum_{i=1}^n p_i}, \end{aligned}$$

which is equivalent to the inequality of the statement. □

**Remark 2.2.** Since  $\frac{A}{M} \geq 0$  in inequality (2.4), we find the inequality between the weighted power means [6, 9],

$$M_r(a, p) \leq M_s(a, p), \quad (2.6)$$

for  $0 < r \leq s$ . The two means are equal if and only if  $a_1 = a_2 = \dots = a_n$ .

**Theorem 2.3.** Let  $p, q > 1$  be real numbers satisfying  $\frac{1}{p} + \frac{1}{q} = 1$ . If  $a_i, b_i > 0$  for all  $i = 1 \dots n$ , then there is the following inequality

$$\frac{A}{M} \leq \left( \sum_{i=1}^n a_i^p \right)^{1/p} \left( \sum_{i=1}^n b_i^q \right)^{1/q} - \sum_{i=1}^n a_i b_i \leq \frac{A}{m}, \quad (2.7)$$

where

$$A = \frac{1}{2pq} \left( \sum_{i=1}^n a_i^p \right)^{1/p} \left( \sum_{i=1}^n b_i^q \right)^{1/q} \sum_{i=1}^n \left( \frac{a_i^p}{\sum_{i=1}^n a_i^p} - \frac{b_i^q}{\sum_{i=1}^n b_i^q} \right)^2,$$

$$m = \min_{i=1,n} \left\{ \frac{\frac{a_i^p}{\sum_{i=1}^n a_i^p}, \frac{b_i^q}{\sum_{i=1}^n b_i^q}}{\frac{a_i^p}{\sum_{i=1}^n a_i^p}, \frac{b_i^q}{\sum_{i=1}^n b_i^q}} \right\} \text{ and } M = \max_{i=1,n} \left\{ \frac{\frac{a_i^p}{\sum_{i=1}^n a_i^p}, \frac{b_i^q}{\sum_{i=1}^n b_i^q}}{\frac{a_i^p}{\sum_{i=1}^n a_i^p}, \frac{b_i^q}{\sum_{i=1}^n b_i^q}} \right\}.$$

*Proof.* By replacing  $\lambda = \frac{1}{p}$ ,  $1 - \lambda = \frac{1}{q}$ ,  $a = \frac{a_i^p}{\sum_{i=1}^n a_i^p}$ ;  $b = \frac{b_i^q}{\sum_{i=1}^n b_i^q}$  in inequality (1.2) we obtain the

relation

$$\begin{aligned} \frac{1}{2pqM} \left( \frac{\frac{a_i^p}{\sum_{i=1}^n a_i^p} - \frac{b_i^q}{\sum_{i=1}^n b_i^q}}{\frac{a_i^p}{\sum_{i=1}^n a_i^p}, \frac{b_i^q}{\sum_{i=1}^n b_i^q}} \right)^2 &\leq \frac{\frac{a_i^p}{\sum_{i=1}^n a_i^p}}{p \sum_{i=1}^n a_i^p} + \frac{\frac{b_i^q}{\sum_{i=1}^n b_i^q}}{q \sum_{i=1}^n b_i^q} - \frac{a_i b_i}{\left( \sum_{i=1}^n a_i^p \right)^{1/p} \left( \sum_{i=1}^n b_i^q \right)^{1/q}} \\ &\leq \frac{1}{2pqm} \left( \frac{\frac{a_i^p}{\sum_{i=1}^n a_i^p} - \frac{b_i^q}{\sum_{i=1}^n b_i^q}}{\frac{a_i^p}{\sum_{i=1}^n a_i^p}, \frac{b_i^q}{\sum_{i=1}^n b_i^q}} \right)^2. \end{aligned}$$

We observe that taking the sum for  $i = 1 \dots n$  we deduce the inequality of the statement.  $\square$

**Remark 2.3.** (a) Hölder's inequality is widely used in the theory of inequalities and has the form [6, 9]:

$$\left( \sum_{i=1}^n a_i^p \right)^{1/p} \left( \sum_{i=1}^n b_i^q \right)^{1/q} \geq \sum_{i=1}^n a_i b_i. \quad (2.8)$$

Because  $\frac{A}{M} \geq 0$  in inequality (2.7), we obtain a proof of Hölder's inequality. It is easy to see that inequality (2.7) is a refinement of Hölder's inequality and contains a reverse inequality for the inequality of Hölder.

(b) For  $p = q = 2$  in inequality (2.7), we have an improvement of Cauchy's inequality

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \geq \left( \sum_{i=1}^n a_i b_i \right)^2, \quad (2.9)$$

given by the inequality

$$\frac{A^2}{M^2} + \frac{2A}{M} \sum_{i=1}^n a_i b_i \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{i=1}^n b_i^2 \right) - \left( \sum_{i=1}^n a_i b_i \right)^2 \leq \frac{A^2}{m^2} + \frac{2A}{m} \sum_{i=1}^n a_i b_i, \quad (2.10)$$

where

$$A = \frac{1}{8} \sqrt{\left(\sum_{i=1}^n a_i^2\right) \left(\sum_{i=1}^n b_i^2\right)} \cdot \sum_{i=1}^n \left( \frac{\frac{a_i^2}{n}}{\sum_{i=1}^n a_i^2} - \frac{\frac{b_i^2}{n}}{\sum_{i=1}^n b_i^2} \right)^2,$$

$$m = \min_{i=1,n} \left\{ \frac{\frac{a_i^2}{n}}{\sum_{i=1}^n a_i^2}, \frac{\frac{b_i^2}{n}}{\sum_{i=1}^n b_i^2} \right\} \text{ and } M = \max_{i=1,n} \left\{ \frac{\frac{a_i^2}{n}}{\sum_{i=1}^n a_i^2}, \frac{\frac{b_i^2}{n}}{\sum_{i=1}^n b_i^2} \right\}.$$

The equality holds for  $\frac{a_1}{b_1} = \dots = \frac{a_n}{b_n}$ .

(c) In [10], O. T. Pop gave Bergström's inequality,

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \dots + x_n)^2}{a_1 + a_2 + \dots + a_n} \quad (2.11)$$

for every  $x_k \in \mathbb{R}$  and  $a_k > 0$ ,  $k \in \{1, 2, \dots, n\}$ . If we make substitutions  $a_i = \frac{x_i}{\sqrt{a_i}}$  and  $b_i = \sqrt{a_i}$ , for all  $i = \{1, 2, \dots, n\}$ , in inequality (2.10) we find a new refinement of Bergström's inequality, which is given as follows

$$\begin{aligned} & \left(\sum_{i=1}^n a_i\right)^{-1} \left( \frac{A^2}{M^2} + \frac{2A}{M} \sum_{i=1}^n |x_i| \right) \\ & \leq \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \dots + \frac{x_n^2}{a_n} - \frac{(|x_1| + |x_2| + \dots + |x_n|)^2}{a_1 + a_2 + \dots + a_n} \\ & \leq \left(\sum_{i=1}^n a_i\right)^{-1} \left( \frac{A^2}{m^2} + \frac{2A}{m} \sum_{i=1}^n |x_i| \right), \end{aligned}$$

where

$$A = \frac{1}{8} \sqrt{\left(\sum_{i=1}^n \frac{x_i^2}{a_i}\right) \left(\sum_{i=1}^n a_i\right)} \cdot \sum_{i=1}^n \left( \frac{\frac{x_i^2}{n}}{a_i \sum_{i=1}^n \frac{x_i^2}{a_i}} - \frac{\frac{a_i}{n}}{\sum_{i=1}^n a_i} \right)^2,$$

$$m = \min_{i=1,n} \left\{ \frac{\frac{x_i^2}{n}}{a_i \sum_{i=1}^n \frac{x_i^2}{a_i}}, \frac{\frac{a_i}{n}}{\sum_{i=1}^n a_i} \right\} \text{ and } M = \max_{i=1,n} \left\{ \frac{\frac{x_i^2}{n}}{a_i \sum_{i=1}^n \frac{x_i^2}{a_i}}, \frac{\frac{a_i}{n}}{\sum_{i=1}^n a_i} \right\}.$$

### 3 APPLICATION TO ARITHMETIC FUNCTIONS

In the theory of the arithmetic functions [1, 8, 11], for positive integer  $n$ , several important functions have been studied. Among these we found  $\sigma_k(n)$ ,  $\tau(n)$ ,  $\sigma_k^*(n)$  and  $\tau^*(n)$ , where  $\sigma_k(n)$  is the sum of  $k$ th powers of the divisors of  $n$ ,  $\tau(n)$  is the number of divisors of  $n$ ,  $\sigma_k^*(n)$  is the sum of  $k$ th powers of the unitary divisors of  $n$  and  $\tau^*(n)$  is the number of unitary divisors of  $n$ , where  $k \geq 0$ .

**Theorem 3.1.** *For  $n \geq 1$  and  $k \geq 0$ , there are the following inequalities*

$$\begin{aligned} \frac{1}{2n\tau(n)} \left[ \sigma_{2k}(n) - \left( \frac{\sigma_k(n)}{\tau(n)} \right)^2 \right] &\leq \frac{\sigma_k(n)}{\tau(n)} - \sqrt{n^k} \\ &\leq \frac{1}{2\tau(n)} \left[ \sigma_{2k}(n) - \left( \frac{\sigma_k(n)}{\tau(n)} \right)^2 \right] \end{aligned} \quad (3.1)$$

and

$$\begin{aligned} \frac{1}{2n\tau^*(n)} \left[ \sigma_{2k}^*(n) - \left( \frac{\sigma_k^*(n)}{\tau^*(n)} \right)^2 \right] &\leq \frac{\sigma_k^*(n)}{\tau^*(n)} - \sqrt{n^k} \\ &\leq \frac{1}{2\tau^*(n)} \left[ \sigma_{2k}^*(n) - \left( \frac{\sigma_k^*(n)}{\tau^*(n)} \right)^2 \right]. \end{aligned} \quad (3.2)$$

*Proof.* If  $d_1, d_2, \dots, d_s$  are the divisors of  $n$ , then we take  $\alpha_i = \frac{1}{s}$  and  $x_i = d_i^k$  in inequality (1.1). Therefore, we have  $m = 1$ ,  $M = n$  and  $s = \tau(n)$ , so inequality (1.1) becomes:

$$\frac{1}{2ns} \sum_{i=1}^s \left( d_i^k - \frac{\sigma_k(n)}{\tau(n)} \right)^2 \leq \frac{\sigma_k(n)}{\tau(n)} - \sqrt{n^k} \leq \frac{1}{2s} \sum_{i=1}^s \left( d_i^k - \frac{\sigma_k(n)}{\tau(n)} \right)^2.$$

Making simple calculations and taking into account that  $\left( \prod_{i=1}^s d_i^k \right)^{1/s} = \left( \prod_{i=1}^s d_i \right)^{\frac{k}{s}} = (n^{\frac{s}{2}})^{\frac{k}{s}} = n^{\frac{k}{2}}$ , we observe that this inequality is equivalent to inequality (3.1). Similarly prove that inequality (3.2) is true. □

**Remark 3.1.** *Inequality (3.2) improves the inequality*

$$\frac{\sigma_k(n)}{\tau(n)} \geq \sqrt{n^k}, \quad (3.3)$$

*which is due to S. Sivaramakrishnan and C. S. Venkataraman [11] and inequality (3.2) improves the inequality*

$$\frac{\sigma_k^*(n)}{\tau^*(n)} \geq \sqrt{n^k},$$

*which is due to J. Sándor and L. Tóth [12, 11].*

## 4 APPLICATIONS TO OPERATORS

In this section, we consider bounded linear operators acting on a complex Hilbert space  $\mathcal{H}$ . If a bounded linear operator  $A$  satisfies  $A = A^*$ , then  $A$  is called a self-adjoint operator. If a self-adjoint operator  $A$  satisfies  $\langle x|A|x \rangle \geq 0$  for any  $|x \rangle \in \mathcal{H}$ , then  $A$  is called a positive operator and denoted by  $A \geq 0$ . In addition,  $A \geq B$  means  $A - B \geq 0$ . We also define operator mean by  $A \sharp_\lambda B \equiv A^{1/2} (A^{-1/2} B A^{-1/2})^\lambda A^{1/2}$  for  $\lambda \in [0, 1]$ , two invertible positive operator  $A$  and  $B$  [7]. Note that we have the relation  $B \sharp_{1-\lambda} A = A \sharp_\lambda B$ .

**Theorem 4.1.** *For  $\lambda \in [0, 1]$ , two invertible positive operator  $A$  and  $B$ , we have the following relations.*

(i) *If  $A \leq B$ , then we have*

$$\begin{aligned} \frac{\lambda(1-\lambda)}{2} (AB^{-1}A - 2A + B) &\leq (1-\lambda)A + \lambda B - A \sharp_\lambda B \\ &\leq \frac{\lambda(1-\lambda)}{2} (BA^{-1}B - 2B + A). \end{aligned} \quad (4.1)$$

(ii) *If  $B \leq A$ , then we have*

$$\begin{aligned} \frac{\lambda(1-\lambda)}{2} (BA^{-1}B - 2B + A) &\leq (1-\lambda)A + \lambda B - A \sharp_\lambda B \\ &\leq \frac{\lambda(1-\lambda)}{2} (AB^{-1}A - 2A + B). \end{aligned} \quad (4.2)$$

*Proof:* We prove (i). Exchanging  $\lambda$  and  $1 - \lambda$  in the inequalities (1.2), we have

$$\frac{\lambda(1-\lambda)}{2b} (a-b)^2 \leq (1-\lambda)a + \lambda b - a^{1-\lambda}b^\lambda \leq \frac{\lambda(1-\lambda)}{2a} (a-b)^2$$

in the case of  $a \leq b$ . Thus we have the inequalities for  $0 < t \leq 1$ :

$$\frac{\lambda(1-\lambda)}{2} (t-1)^2 \leq (1-\lambda)t + \lambda - t^{1-\lambda} \leq \frac{\lambda(1-\lambda)}{2} \left( \sqrt{t} - \frac{1}{\sqrt{t}} \right)^2$$

putting  $t \equiv \frac{a}{b}$ . Thus we have for  $0 < T \leq I$ ,

$$\frac{\lambda(1-\lambda)}{2} (T-1)^2 \leq (1-\lambda)T + \lambda - T^{1-\lambda} \leq \frac{\lambda(1-\lambda)}{2} (T^{1/2} - T^{-1/2})^2$$

by standard operational calculus. Putting  $T = B^{-1/2}AB^{-1/2}$  and then multiplying  $B^{1/2}$  from the both sides, we obtain the desired results. (ii) can be proven by the similar way to the proof of (i).  $\square$



**Remark 4.1.** We have  $AB^{-1}A - 2A + B = A^{1/2} \left( A^{1/2}B^{-1}A^{1/2} + (A^{1/2}B^{-1}A^{1/2})^{-1} - 2I \right) A^{1/2} \geq 0$ , because we have  $u + u^{-1} - 2 = \frac{(u-1)^2}{u} \geq 0$  for scalar  $u \geq 0$ . By the similar way, we have  $BA^{-1}B - 2B + A \geq 0$ . Thus under the condition  $A \leq B$  or  $B \leq A$ , the inequalities in Theorem 4.1 improve the second inequality of the following inequalities (See [5, 3] for example):

$$\{(1-\lambda)A^{-1} + \lambda B^{-1}\}^{-1} \leq A\sharp_{\lambda}B \leq (1-\lambda)A + \lambda B \quad (4.3)$$

**Corollary 4.1.** For  $\lambda \in (0, 1)$ , two invertible positive operator  $A$  and  $B$ , we have the following relations.

(i) If  $A \leq B$ , then we have

$$\begin{aligned} & A\sharp_{\lambda}B - (A\sharp_{\lambda}B) \left\{ \frac{2}{\lambda(1-\lambda)} (B^{-1}AB^{-1} - 2B^{-1} + A^{-1})^{-1} + A\sharp_{\lambda}B \right\} (A\sharp_{\lambda}B) \\ & \leq \{(1-\lambda)A^{-1} + \lambda B^{-1}\}^{-1} \\ & \leq A\sharp_{\lambda}B - (A\sharp_{\lambda}B) \left\{ \frac{2}{\lambda(1-\lambda)} (A^{-1}BA^{-1} - 2A^{-1} + B^{-1})^{-1} + A\sharp_{\lambda}B \right\} (A\sharp_{\lambda}B) \end{aligned} \quad (4.4)$$

(ii) If  $B \leq A$ , then we have

$$\begin{aligned} & A\sharp_{\lambda}B - (A\sharp_{\lambda}B) \left\{ \frac{2}{\lambda(1-\lambda)} (A^{-1}BA^{-1} - 2A^{-1} + B^{-1})^{-1} + A\sharp_{\lambda}B \right\} (A\sharp_{\lambda}B) \\ & \leq \{(1-\lambda)A^{-1} + \lambda B^{-1}\}^{-1} \\ & \leq A\sharp_{\lambda}B - (A\sharp_{\lambda}B) \left\{ \frac{2}{\lambda(1-\lambda)} (B^{-1}AB^{-1} - 2B^{-1} + A^{-1})^{-1} + A\sharp_{\lambda}B \right\} (A\sharp_{\lambda}B) \end{aligned} \quad (4.5)$$

*Proof:* Replacing  $A$  and  $B$  by  $A^{-1}$  and  $B^{-1}$  in the inequalities (4.1), respectively and taking the inverse of both sides, then we have the inequalities (4.4), using  $(A^{-1}\sharp_{\lambda}B^{-1})^{-1} = A\sharp_{\lambda}B$  and

$$(X^{-1} + Y^{-1})^{-1} = X^{-1} - X^{-1}(X^{-1} + Y^{-1})^{-1}X^{-1}$$

for invertible positive operators  $X$  and  $Y$ . The inequalities (4.5) can be proven by the similar way to the inequalities (4.4). □

**Remark 4.2.** Since  $(A^{-1}BA^{-1} - 2A^{-1} + B^{-1})^{-1} \geq 0$ ,  $(B^{-1}AB^{-1} - 2B^{-1} + A^{-1})^{-1} \geq 0$  and  $A\sharp_{\lambda}B \geq 0$ , then the right hand side of the inequalities (4.4) and (4.5) are further bounded from the above by  $A\sharp_{\lambda}B$ . Therefore two inequalities (4.4) and (4.5) improve the first inequality of the inequalities (4.3) under the condition  $A \leq B$  or  $B \leq A$ .

**Corollary 4.2.** If  $0 < A \leq B$ , then we have

$$3(A - B) + BA^{-1}B - AB^{-1}A \geq 0. \quad (4.6)$$

The inequality (4.6) corresponds to the following relation:

$$0 < a \leq b \Rightarrow (b - a)^3 \geq 0$$

in the commutative case. The inequality (4.6) can be directly proven by applying the standard operational calculus to the scalar inequality  $(t - 1)^3 \leq 0$  for  $0 < t \leq 1$ .

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